the set is called a *substitution group* of order $g$. The number of different elements or letters that occur in all of the substitutions of the group is said to be the *degree* of the group. Hence the given six substitutions constitute a substitution group of order 6 and of degree 3. When a group is contained in a larger group it is said to be a *subgroup* of the larger group; *e. g.* the group of order 3 and degree 3 whose substitutions are 1, *abc*, *acb* is a subgroup of the given group of order 6. This group of order 6 contains four other subgroups, three of order 2, and one of order 1. It may be observed that the subgroup, identity, occurs in every group.

The substitution *abc* means that $a$ is replaced by $b$, and then $b$ by $c$, and finally $c$ by $a$. If we suppose that these three letters are placed in the given order on the circumference of a circle at intervals of 120° the given substitution is equivalent to a positive rotation of this circle through 120° degrees. Hence such a substitution is called a cycle or a circular substitution. It is evident that the given notation is not unique, for $abc=bca=cab$. In general, if a circular substitution contains $n$ elements it may be written in $n$ ways. This indefiniteness is generally avoided by beginning with the first letter of the alphabet that occurs in the substitution. If this is done the notation becomes unique.

Any substitution whatever is the operation by means of which we may derive a particular permutation from a given arrangement of the elements involved in the substitution, and every rearrangement of the elements of a given permutation leads to a substitution in those elements. Hence we observe that any substitution consists either of a single cycle or of a series of cycles such that no two of them have a common element: *e. g.* the permutation *abcdefghi* is obtained from the permutation *cabedifgh* by means of the substitution *abc.de.fghi*, the periods being used to separate the complete cycles. Since the operations indicated by the different cycles may be performed independently of each other and in any order the given periods may also be interpreted as indicating multiplication.

According to the given notation *any series of letters or elements may be regarded as a substitution provided no letter occurs more than once in the series, and these letters are either not separated by any marks or they are divided into sets of two or more by means of periods.*

Hence there are two methods by means of which we can obtain all the possible substitutions that can be formed with a given number ($n$) of letters. By the first method we write down all the possible permutations of these $n$ letters and find the substitutions by means of which we can obtain all the $n!$ permutations from any given one of them. By the second method we write down all the possible different substitutions that actually involve the $n$ letters, then those that involve any combination of $n-1$, $n-2$, .... 3, 2 of them. The sum of these will, of course, be $n!$

The beginner would do well if he would find the twenty-four possible substitutions whose degree does not exceed four by means of each of these two methods. It may be remarked that there are just $(n-1)!$ circular substitutions that contain $n$ given letters.

# INTEGRATION BY ELLIPTIC INTEGRALS.

By GEORGE B. McCLELLAN ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

We will first expand the general expression

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}.$$

Let $2\cos\varphi=s+s^{-1}$. $\therefore (1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}=(1+e^2-es-es^{-1})^{-\frac{1}{2}(2m+1)}$

$$=(1-es)^{-\frac{1}{2}(2m+1)}(1-es^{-1})^{-\frac{1}{2}(2m+1)}=\Big(1+\frac{2m+1}{2}.es+\frac{(2m+1)(2m+3)}{2^2.2!}.e^2s^2$$

$$+\frac{(2m+1)(2m+3)(2m+5)}{2^3.3!}.e^3s^3+\frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4.4!}.e^4s^4+\ldots\ldots\Big)$$

$$\Big(1+\frac{2m+1}{2}.es^{-1}+\frac{(2m+1)(2m+3)}{2^2.2!}e^2s^{-2}+\frac{(2m+1)(2m+3)(2m+5)}{2^3.3!}e^3s^{-3}$$

$$+\frac{(2m+1)(2m+3)(2m+5)(2m+7)}{2^4.4!}.e^4s^{-4}+\ldots\ldots\Big)=\Big[1+\Big(\frac{2m+1}{2}.e\Big)^2$$

$$+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{e^2}{2!}\Big)^2+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{2m+5}{2}.\frac{e^3}{3!}\Big)^2+\ldots\ldots\Big]$$

$$+2\Big[\frac{2m+1}{2}.e+\Big(\frac{2m+1}{2}\Big)^2.\frac{2m+3}{2}.\frac{e^3}{2!}+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}\Big)^2.\frac{2m+5}{2}.\frac{e^5}{2!.3!}$$

$$+\ldots\ldots\Big]\Big[\frac{s+s^{-1}}{2}\Big]+2\Big[\frac{2m+1}{2}.\frac{2m+3}{2}.\frac{e^2}{2!}+\Big(\frac{2m+1}{2}\Big)^2.\frac{2m+3}{2}.\frac{2m+5}{2}.\frac{e^4}{3!}$$

$$+\Big(\frac{2m+1}{2}.\frac{2m+3}{2}\Big)^2.\frac{2m+5}{2}.\frac{2m+7}{2}.\frac{e^6}{2!.4!}+\ldots\ldots\Big]\Big[\frac{e^2+e^{-2}}{2}\Big]+\ldots\ldots(A).$$

$$\therefore (1+e^2-2e\cos\varphi)^{-\frac{1}{2}(2m+1)}=\tfrac{1}{2}P_0+P_1\cos\varphi+P_2\cos2\varphi+P_3\cos3\varphi+\ldots\ldots$$
$$+P_n\cos n\varphi.$$

When $m=0$, 1, 2, 3, etc., we get

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\tfrac{1}{2}A_0+A_1\cos\varphi+A_2\cos2\varphi+A_3\cos3\varphi+\ldots\ldots \quad \ldots\ldots\ldots(B).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\tfrac{1}{2}B_0+B_1\cos\varphi+B_2\cos2\varphi+B_3\cos3\varphi+\ldots\ldots\ldots\ldots\ldots\ldots(C).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\tfrac{1}{2}C_0+C_1\cos\varphi+C_2\cos2\varphi+C_3\cos3\varphi+\ldots\ldots\ldots\ldots\ldots\ldots(D).$$

$$(1+e^2-2e\cos\varphi)^{-\frac{1}{2}}=\tfrac{1}{2}D_0+D_1\cos\varphi+D_2\cos2\varphi+D_3\cos3\varphi+\ldots\ldots\ldots\ldots\ldots\ldots(E).$$

Let $\sin(\theta-\varphi)=e\sin\theta$, then $\tan\theta=\sin\theta/(\cos\varphi-e)$.

$$\therefore \cos(\theta-\varphi)\left(1-\frac{d\varphi}{d\theta}\right)=e\cos\theta.$$

$$\therefore \frac{d\varphi}{d\theta}=\frac{\cos(\theta-\varphi)-e\cos\theta}{\cos(\theta-\varphi)}=\frac{\sin\theta\sin\varphi+\cos\theta(\cos\varphi-e)}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore \frac{d\varphi}{d\theta}=\frac{(\cos^2\theta\operatorname{cosec}\theta+\sin\theta)\sin\varphi}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{\sin\varphi\operatorname{cosec}\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\sqrt{\frac{(\cos\varphi-e)^2+\sin^2\varphi}{1-e^2\sin^2 b}}=\sqrt{\frac{1+e^2-2e\cos\varphi}{1-e^2\sin^2\theta}}\ldots\ldots\ldots(1_0).$$

Also $\sqrt{(1+e^2-2e\cos\varphi)}=\sqrt{(1-e^2\sin^2\theta)}-e\cos\theta.$

$$\therefore \cos\varphi=e\sin^2\theta+\cos\theta\sqrt{(1-e^2\sin^2\theta)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2_0).$$

$$\cos2\varphi=4e^2\sin^4\theta+4e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+1-2(1+e^2)\sin^2\theta\ldots\ldots\ldots(3_0).$$

$$\cos3\varphi=4e^3\cos^6\theta+12e^2\sin^4\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+12e\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)$$
$$+4\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}-3e\sin^2\theta-3\cos\theta\sqrt{(1-e^2\sin^2\theta)}\ldots\ldots\ldots\ldots(4_0).$$

$$\cos4\varphi=1-8e^2\sin^4\theta-16e\sin^2\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}-8\cos^2\theta(1-e^2\sin^2\theta)$$
$$+8e^4\sin^8\theta+32e^3\sin^6\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}+48e^2\sin^4\cos^2\theta(1-e^2\sin^2\theta)$$
$$+8\cos^4\theta(1-e^2\sin^2\theta)^2+32e\sin^2\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}\ldots\ldots\ldots\ldots\ldots(5_0).$$

$$\cos5\varphi=16e^5\sin^{10}\theta+80e^4\sin^8\cos\theta\sqrt{(1-e^2\sin^2\theta)}+160e^3\sin^6\theta\cos^2\theta(1-e^2\sin^2\theta)$$
$$+160e^2\sin^4\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}+80e\sin^2\theta\cos^4\theta(1-e^2\sin^2\theta)^2$$
$$+16\cos^5\theta(1-e^2\sin^2\theta)^{\frac{5}{2}}-20e^3\sin^6\theta-60e^2\sin^4\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}$$
$$-60\sin^2\theta\cos^2\theta(1-e^2\sin^2\theta)-20\cos^3\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}$$
$$+5e\sin^2\theta+5\cos\theta\sqrt{(1-e^2\sin^2\theta)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6_0),$$

$$\cos 6\varphi = 32e^6\sin^{12}\theta + 192e^5\sin^{10}\theta\cos\theta_1/(1-e^2\sin^2\theta) + 480e^4\sin^8\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$+640e^3\sin^6\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{1}{2}} + 480e^2\sin^4\theta\cos^4\theta(1-e^2\sin^2\theta)^2$$

$$+192e\sin^2\theta\cos^5\theta(1-e^2\sin^2\theta)^{\frac{1}{2}} + 32\cos^6\theta(1-e^2\sin^2\theta)^3 - 48e^4\sin^8\theta$$

$$-192e^3\sin^6\theta\cos\theta_1/(1-e^2\sin^2\theta) - 288e^2\sin^4\theta\cos^2\theta(1-e^2\sin^2\theta)$$

$$-192e\sin^2\theta\cos^3\theta(1-e^2\sin^2\theta)^{\frac{1}{2}} - 48\cos^4\theta(1-e^2\sin^2\theta)^2 + 18e^2\sin^4\theta$$

$$+18\cos^2\theta(1-e^2\sin^2\theta) + 36e\sin^2\theta\cos b_1/(1-e^2\sin^2\theta) - 1 \quad \ldots\ldots\ldots\ldots (7_0).$$

Writing $(B)$ in the following form :

$$(1+e^2-es-es^{-1})^{-\frac{1}{2}} = \tfrac{1}{2}A_0 + \tfrac{1}{2}A_1(s+s)^{-1}) + \ldots\ldots + \tfrac{1}{2}A_n(s^n+s^{n-1}) + \ldots\ldots\ldots (8_0).$$

Differentiating $(8_0)$ we get,

$$e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{3}{2}} = A_1(1-s^{-2}) + 2A_2(s-s^{-3})$$

$$+3A_3(s^2-s^{-4}) + \ldots\ldots + nA_n(s^{n-1}-s^{-(n+1)}) + \ldots\ldots (9_0).$$

From $(8_0)$ and $(9_0)$ we get,

$$\tfrac{1}{2}e(1-s^{-2})[A_0 + A_1(s+s^{-1}) + A_2(s^2+s^{-2}) + \ldots\ldots + A_n(s^n+s^{-n}) + \ldots\ldots]$$

$$= (1+e^2-es-es^{-1})[A_1(1-s^{-2}) + 2A_2(s-s^{-3}) + 3A_3(s^2-s^{-4})$$

$$+ \ldots\ldots + nA_n(s^{n-1}-s^{-(n+1)}) + \ldots\ldots].$$

Equating coefficients of $s^n$ we get,

$$\tfrac{1}{2}e(A_n - A_{n+2}) = (n+1)(1+e^2)A_{n+1} - e[nA_n + (n+2)A_{n+2}].$$

$$\therefore A_{n+2} = \frac{2(n+1)}{2n+3}\cdot\frac{1+e^2}{e}A_{n+1} - \frac{2n+1}{2n+3}A_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (10_0).$$

$\therefore$ When $A_n$ and $A_{n+1}$ are known we can easily find $A_{n+2}$.

Multiplying $(B)$ by $\cos\varphi$ we get,

$$\frac{\cos\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = \tfrac{1}{2}A_0\cos\varphi + \tfrac{1}{2}A_1(1+\cos 2\varphi) + \tfrac{1}{2}A_2(\cos\varphi + \cos 3\varphi) + \ldots\ldots (11_0).$$

Also $\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = F(e, \tfrac{1}{2}\pi)$ ..... .....(1).

$$\int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,d\theta = E(e, \tfrac{1}{2}\pi) \text{ ..... .....} (2).$$

Integrating both sides of $(B)$ and $(11_0)$ between the limits $2\pi$ and 0, we get with the aid of $(1_0)$, $(2_0)$, (1), (2), the following:

$$\pi A_0 = \int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} = 4F(e, \tfrac{1}{2}\pi).$$

$$\therefore A_0 = (4/\pi)F(e, \tfrac{1}{2}\pi) \text{ ..... .....} (12_0).$$

$$\pi A_1 = \int_0^{2\pi} \frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{1}{2}}} = 4e\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta d\theta}{\sqrt{1-e^2\sin^2\theta}} + \int_0^{2\pi} \cos\theta d\theta$$

$$= \frac{4}{e}\left[\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-e^2\sin^2\theta}} - \int_0^{\frac{1}{2}\pi} \sqrt{1-e^2\sin^2\theta}\,d\theta\right] = \frac{4}{e}[F(e, \tfrac{1}{2}\pi) - E(e, \tfrac{1}{2}\pi)].$$

$$\therefore A_1 = (4/\pi e)[F(e, \tfrac{1}{2}\pi) - E(e, \tfrac{1}{2}\pi)].$$

[To be Continued.]

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## AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION."

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from October Number.]

### CHAPTER III.

MULTIPLICATION OF EXTENSIVE QUANTITIES. DIFFERENT KINDS OF MULTIPLICATION.

22. In the multiplication of extensive quantities expressed in terms of units, it is assumed that the distributive law holds, and that numerical coefficients may be treated as in elementary algebra (16).

Thus if $a=\Sigma\alpha_r e_r$ and $b=\Sigma\beta_s e_s$ are two extensive quantities in which $\alpha$ and $\beta$ are numbers and the $e$'s are extensive *units*, we may write

$$ab=[\Sigma\alpha_r e_r . \Sigma\beta_s e_s]=\Sigma\alpha_r\beta_s[e_r e_s],$$

that is to say, in the result each term of the multiplier is multiplied into every term of the multiplicand, and the partial products are added.

Notice that the law is assumed to hold only when the two factors are sums of *units*. In the theorems which follow it is shown that the same law applies when the factors are sums of *quantities*.

23. Before going on to the proofs of these theorems, we will illustrate the question involved by an example. (See Art. 9).

Let

$$\rho_1=r_{11}e_1+r_{12}e_2+r_{13}e_3+\ldots\ldots$$

$$\rho_2=r_{21}e_1+r_{22}e_2+r_{23}e_3+\ldots\ldots$$

Also, let

$$a=\alpha_1\rho_1+\alpha_2\rho_2+\ldots\ldots$$

$$b=\beta_1\rho_1+\beta_2\rho_2+\ldots\ldots$$

It is to be shown, then, that if $\rho_1\rho_2=\Sigma r_{1r}r_{2s}[e_r e_s]$, $ab=\Sigma\alpha_r\beta_s[\rho_r\rho_s]$.

The proof will be based on the definitions laid down and the theorems proved in the last chapter.

Remark.—It should be kept in mind in Articles 24—30 that $\alpha$, $\beta$, ..... denote numbers, the $e$'s denote extensive *units* (11), and $a$, $b$, ..... denote extensive *quantities* (12). Square brackets are used to indicate that the quantities inside are extensive quantities whose product is required.

24. *To show that* $[\Sigma\alpha_r e_r . b]=\Sigma\alpha_r[e_r b]$, *i. e.* to show that in multiplying $\Sigma\alpha_r e_r$ by $b$ each term of $\Sigma\alpha_r e_r$ is multiplied by $b$.

Proof.—Let $b=\Sigma\beta_s e_s$. Then

$$[\Sigma\alpha_r e_r . b]\equiv[\Sigma\alpha_r e_r . \Sigma\beta_s e_s]=\Sigma\alpha_r\beta_s[e_r e_s] \quad (22)$$

$$=\Sigma\alpha_1\beta_s[e_1 e_s]+\Sigma\alpha_2\beta_s[e_2 e_s]+\ldots\ldots \quad (14)$$

$$=\alpha_1\Sigma\beta_s[e_1 e_s]+\alpha_2\Sigma\beta_s[e_2 e_s]+\ldots\ldots \quad (15)$$

$$=\alpha_1[e_1.\Sigma\beta_s e_s]+\alpha_2[e_2.\Sigma\beta_s e_s]+ \ldots\ldots\ldots\ldots\ldots\ldots (22)$$

$$=\alpha_1[e_1.b]+\alpha_2[e_2.b]+\ldots\ldots=\Sigma\alpha_r[e_r b].$$

25. *To show that* $[(a+b+\ldots\ldots)p]=[ap]+[bp]+\ldots\ldots$

$$[p(a+b+\ldots\ldots)]=[pa]+[pb]+\ldots\ldots$$

PROOF.—Let $a=\Sigma\alpha_r e_r$, $b=\Sigma\beta_r e_r\ldots\ldots$ Then

$$[(a+b+\ldots\ldots)p]=[(\Sigma\alpha_r e_r+\Sigma\beta_r e_r+\ldots\ldots)p]=[\Sigma(\alpha_r+\beta_r+\ldots\ldots)e_r.p] \quad (14)$$

$$=\Sigma(\alpha_r+\beta_r+\ldots\ldots)[e_r p] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24)$$

$$=\Sigma[\alpha_r e_r.p]+\Sigma[\beta_r e_r.p]+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (14, 24)$$

$$[ap]+[bp]+\ldots\ldots$$

26. *To show that* $[(\alpha a)b]=\alpha[ab]$, and $[b(\alpha a)]=\alpha[ba]$.

PROOF.—Let $a=\Sigma\alpha_r e_r$. Then

$$[(\alpha a)b]\equiv[(\alpha\Sigma\alpha_r e_r)b]=[\Sigma\alpha\alpha_r e_r)b]\ldots\ldots \ldots\ldots\ldots\ldots (15)$$

$$=\Sigma\alpha\alpha_r[e_r b] \ (24) =\alpha[\Sigma\alpha_r e_r.b] \ (16, 24) =\alpha[ab].$$

The other formula is obtained by making $b$ the first factor in the above proof.

27. *To show that* $[(\alpha a+\beta b+\ldots\ldots)p]=\alpha[ap]+\beta[bp]+\ldots\ldots$

and $[p(\alpha a+\beta b+\ldots\ldots)]=\alpha[pa]+\beta[pb]+\ldots\ldots$

PROOF.—$[(\alpha a+\beta b+\ldots\ldots)p]=[(\alpha a)p]+[(\beta b)p]+ \ldots\ldots\ldots\ldots\ldots (25)$

$$=\alpha[ap]+\beta[bp]+ \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

28. We are now in position to show that the distributive law holds when the two factors are sums of multiples of extensive *quantities* as well as when they are sums of multiples of units.

PROOF.—$[\Sigma\alpha_r a_r.\Sigma\beta_s b_s]=\Sigma\alpha_r[a_r.\Sigma\beta_s b_s] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27)$

$$=\Sigma\alpha_r(\Sigma\beta_s[a_r b_s]) \ (27) =\Sigma\alpha_r\beta_s[a_r b_s] \ldots\ldots\ldots (16)$$

This theorem holds also for any number of factors, as can be shown by mathematical induction.

29. Let us denote a product containing a number of factors, $a$, $b$, $\ldots\ldots$

by $P_{a,\,b\ldots\ldots}$ In such a product suppose a factor $p$ equals $qa+rb+\ldots\ldots$, where $q$, $r$, ..... are numbers; to show that

$$P_{qa+rb+\ldots\ldots}=q.P_{a}+r.P_{b+\ldots\ldots}$$

PROOF.—However the product may be made up, we can always regard $p$ as combined with another factor, then this product with other factors in turn. In each of the multiplications in which $p$ enters as one factor Art. 27 applies.

30. *To show that* $P_{qa,\,rb,\,sc,\,\ldots\ldots}=qrs\ldots\ldots P_{a,\,b,\,c\ldots\ldots}$

PROOF.—By 29, $P_{qa}=q.P_{a}$. Then

$$P_{qa,\,rb,\,sc\ldots\ldots}=q.P_{a,\,rb,\,sc\ldots\ldots}=qrs\ldots\ldots\ P_{a,\,b,\,c\ldots\ldots}$$

It evidently follows that $P_{qa,\,ra}=P_{ra,\,qa}$.

31. *Different Kinds of Multiplication.*—Different kinds of multiplication are obtained by laying down different laws for simplifying a distributed product.

To illustrate:—

$$(1)\ (\alpha_1 e_1+\alpha_2 e_2)(\beta_1 e_1+\beta_2 e_2)=\alpha_1\beta_1 e_1^2+\alpha_1\beta_2 e_1 e_2+\alpha_2\beta_1 e_2 e_1+\alpha_2\beta_2 e_2^2.$$

In ordinary algebra the result is simplified by supposing $e_1 e_2=e_2 e_1$. The result in this way becomes

$$\alpha_1\beta_1 e_1^2+(\alpha_1\beta_2+\alpha_2\beta_1)e_1 e_2+\alpha_2\beta_2 e_2^2,$$

or, say, $m_1 e_1^2+m_2 e_1 e_2++m_3 e_2^2$, where the $m$'s are numerical coefficients.

(2) Similarly, writing three factors, we get,

$$(\alpha_1 e_1+\alpha_2 e_2)(\beta_1 e_1+\beta_2 e_2)(\gamma_1 e_1+\gamma_2 e_2)=m_1 e_1^3+m_2 e_1^2 e_2+m_3 e_1 e_2^2+m_4 e_2^3$$

by supposing, as in ordinary algebra, that

$$\begin{cases} e_1 e_1 e_2=e_1 e_2 e_1=e_2 e_1 e_1 \\ e_1 e_2 e_2=e_2 e_1 e_2=e_2 e_2 e_1 \end{cases}$$

Here the assumed law of simplification reduces a product which would otherwise have eight terms to one of four.

(3) The law of simplification in quaternions (which is a branch of mathematics using a certain kind of extensive quantities) may be seen by multiplying together two factors of three terms each.

Thus, $(\alpha_1 e_1+\alpha_2 e_2+\alpha_3 e_3)(\beta_1 e_1+\beta_2 e_2+\beta_3 e_3)=$

$$-(\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)+(\alpha_1\beta_2-\alpha_2\beta_1)e_3+(\alpha_2\beta_3-\alpha_3\beta_2)e_1+(\alpha_3\beta_1-\alpha_1\beta_3)e_2,$$

by supposing $e_1^2=e_2^2=e_3^2=-1$, $e_1 e_2=e_3$, $e_2 e_3=e_1$, and $e_3 e_1=e_2$.

32. DEFINITION.—A multiplication is said to be *linear* when the same laws of simplication of the distributed product continue to hold when numerically derived quantities (10) replace the given units.

33. *To show that there are but four kinds of linear multiplication.*

Let

$$(a) \qquad \Sigma\alpha_{rs}[e_re_s]=0$$

express a simplification law in the product $\Sigma\alpha_r\beta_s[e_re_s]$.

Let, now, $e_r$ be replaced by $\Sigma x_{r,u}\,e_u$ and $e_s$ by $\Sigma x_{sv}e_v$. We thus get

$$\Sigma\alpha_{rs}[\Sigma x_{ru}e_u.\Sigma x_{sv}e_v]=0,$$

whence, $\Sigma\alpha_{rs}\Sigma x_{ru}x_{sv}[e_ue_v]=0$, (28).

or, $\Sigma\alpha_{rs}x_{ru}x_{sv}[e_ue_v]=0$. (16)

This equation is symmetrical in $r$ and $s$ and $u$ and $v$ and evidently will continue to hold true when these letters are interchanged. This gives

$$\Sigma\alpha_{sr}x_{sv}x_{ru}[e_ve_u]=0.$$

Adding the last two equations, we have

$$(b) \qquad \Sigma x_{ru}x_{sv}\{\alpha_{rs}[e_ue_v]+\alpha_{sr}[e_ve_u]\}=0.$$

Equation ($b$) may also be gotten by multiplying $\Sigma\alpha_{rs}x_{ru}x_{sv}[e_ue_v]$ by 2 and arranging the result with reference to equal coefficients, ($x_{ru}x_{sv}$). This derivation shows ($b$) to be a necessary, and not, as might appear, an arbitrary inference from the given equation.

Now from the nature of the case the coefficients $x_{ru}, x_{sv}$ must be capable of having any values, as would the $x$'s in Articles 6—9. If we assume that the products $x_{ru}x_{sv}$ are arbitrary, then from the theory of equations we have

$$(c) \qquad \alpha_{rs}[e_ue_v]+\alpha_{sr}[e_ve_u]=0,$$

true for all values of $r$ and $s$ and $u$ and $v$.*

If we put $u=v$ in ($c$) we get

$$(d) \qquad (\alpha_{rs}+\alpha_{sr})[e_ue_u]=0.$$

This equation is satisfied either by assuming (1) $\alpha_{rs}+\alpha_{sr}=0$, *i. e.*, $\alpha_{rs}=-\alpha_{sr}$; or, by assuming (2), $[e_ue_v]=0$.

*Grassmann's derivation of ($c$) does not assume that the products $x_{ru}x_{sv}$ are arbitrary. The writer gave another demonstration which does not assume this before the Mathematical Section of the American Association for the Advancement of Science, 1899 meeting. Though somewhat simpler than Grassmann's proof, it would add materially to the length of this article.

(1) If $\alpha_{rs}=-\alpha_{sr}$, and we make this substitution in ($c$), there results

$$\alpha_{rs}\{[e_ue_v]-[e_ve_u]\}=0.$$

In this equation either $\alpha_{rs}=0$, or $[e_ue_v]=[e_ve_u]$. If $\alpha_{rs}=0$, all the coefficients reduce to zero, and equation ($a$) vanishes identically, which is contrary to hypothesis. If

$$(e) \qquad [e_ue_v]-[e_ve_u]=0, \text{ or } [e_ue_v]=[e_ve_u],$$

we have the law for a form of multiplication of extensive quantities which is analogous to ordinary multiplication in algebra. See Art. 31, (1).

(2) If we say $[e_ue_u]=0$, it is equivalent to making in equation ($a$) $\alpha_{rr}=1$, and all the other coefficients equal to 0. Making this substitution in ($c$), we get

$$(f) \qquad [e_ue_v]+[e_ve_u]=0,$$

which implies $[e_ue_u]=0$, as may be seen by making $u=v$ in ($f$).

We have seen that equations ($e$) and ($f$) are necessary conditions in order that a multiplication may be linear. That they are sufficient conditions may be seen as follows: If we start with

$$(a) \qquad [e_re_s]\pm[e_se_r]=0,$$

and substitute as above we get

$$(c) \qquad [e_re_s]\pm[e_se_r]=0.$$

Hence, by definition, (32) ($e$) and ($f$) give linear multiplications.

We have then four kinds of linear multiplication, viz:

1st. That in which there are no simplifying equations.

2nd. That in which all the coefficients in ($a$) are identically 0.

3d. That whose law of simplification is $[e_ue_v]=[e_ve_u]$.

4th. That whose law of simplification is $[e_ue_v]=-[e_ve_u]$.

As between ($e$) and ($f$), the latter gives the simpler species of multiplication. To see this let us take the distributed product in (1) Art. 31. Equation ($e$) reduces the product, as we saw in that Article to three terms. But taking ($f$) as the simplification law, we get a single term, viz., $(\alpha_1\beta_2-\alpha_2\beta_1)e_1e_2$.

34. The *Ausdehnungslehre* concerns itself very largely with the operation of multiplication, especially with what is called *combination* multiplication. This multiplication is based on the law described in the last Article, viz., $e_re_s=-e_se_r$, which also implies $e_r\,e_r=0$.

[To be Continued.]

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ARITHMETIC.

112. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$4.\frac{.\dot{2}9\dot{7}}{1.00\dot{3}}$. The selling price is $\$6.\frac{1000}{33337}$. What is the gain % ?

II. Solution by JOHN M. HOWIE, Professor of Mathematics, The Nebraska State Normal, Peru, Neb.

In the solution of the above in the August-September MONTHLY by Professors Schmitt and Zerr, the following statements are made :

$.\dot{2}9\dot{7}=\frac{297}{999}=\frac{11}{37}$ ; $1.00\dot{3}=1\frac{1}{300}=\frac{301}{300}$. $\frac{11}{37}\div\frac{301}{300}=\frac{3300}{11137}$ ; $.\frac{3300}{11137}=\frac{3300}{11137}$.

It seems to me the statements are inconsistent.

If $.\frac{3300}{11137}=\frac{300}{11137}$ then also $.\dot{2}9\dot{7}=.\frac{297}{999}=.\frac{11}{37}=\frac{11}{370}$.

But if $.\dot{2}9\dot{7}=\frac{297}{999}=\frac{11}{37}$, then $.\frac{3300}{11137}=\frac{3300}{11137}$.

The latter I think to be correct.

The following solution seems to me to be correct :

$.\dot{2}9\dot{7}=.\frac{297}{999}=.\frac{11}{37}=\frac{11}{37}$ ; $1.00\dot{3}=1\frac{1}{300}=\frac{301}{300}$. $\frac{11}{37}\div\frac{301}{300}=\frac{3300}{11137}$ ; $.\frac{3300}{11137}=\frac{3300}{11137}$.

$\therefore\ \$4.\frac{.\dot{2}9\dot{7}}{1.00\dot{3}}=\$4\frac{3300}{11137}=\$\frac{47848}{11137}$=cost price.

$\$6.\frac{1000}{33337}=\$6\frac{1000}{33337}=\$\frac{201022}{33337}$=selling price.

$\therefore\ \$\frac{201022}{33337}-\$\frac{47848}{11137}=\$\frac{1738266574}{1003443437}$=gain.

$\frac{1738266574}{1003443437}\div\frac{47848}{11137}=.401\frac{901935}{5388881}=401\frac{901935}{5388881}\%$=gain per cent.

117. Proposed by MARCUS BAKER, U.S. Coast and Geodetic Survey, 1905 Sixteenth St., Washington, D.C.

A landed man two daughters had,
And both were very fair;
He gave to each a piece of land,
One round, the other square.

At twenty shillings an acre, just,
Each piece its value had;
The shillings that did compass each,
For it exactly paid.

If 'cross a shilling be an inch,
(As it is, very near),
Which had the larger portion, she
That had the round or square?

Also, how many acres did each receive?
[Does anyone know the history of this problem?]

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and ELMER SCHUYLER, Reading, Pa.

Let $r$=radius of circle, $x$=side of square, both in rods.

In this solution we regard the shillings on the perimeter of the circle as tangent at the extremities of their diameters.

$$\therefore \frac{20\pi r^2}{160}=2\pi r\times 16\tfrac{1}{2}\times 12.$$

$$\therefore r=3068 \text{ rods}; \frac{\pi r^2}{160}=184816.87224 \text{ acres}.$$

$$\frac{20x^2}{160}=4x\times 16\tfrac{1}{2}\times 12.$$

$$\therefore x=6336 \text{ rods}; \frac{x^2}{160}=250905.6 \text{ acres}.$$

II. Solution by D. G. DORRANCE, Jr., Camden, N. Y.

(1) Let $2\pi R=x$=number of inches in circumference of circular parcel, and (2) $\pi R^2=x/20$ acres$=313632x$ square inches.

From (1), $R=\frac{x}{2\pi}$, or $R^2=\frac{x^2}{4\pi^2}$.

From (2), $R^2=\frac{313632x}{\pi}$. Then $\frac{x^2}{4\pi^2}=\frac{313632x}{\pi}$.

Whence $x=313632(4\pi)=3941225.1648$ inches in circumference.
$x/20=197061.25824$, number of acres in circular parcel.

(1) Let $4y=x$=number of inches around the square.
(2) Let $y^2=x/20$ acres$=313632x$ square inches.
From (1), $y=x/4$, or $y^2=x^2/16$.
From (2), $y^2=313532x$.
Then $x^2/16=313632x$.
Whence $x=5018112$. $x/20=250905.6$=number of acres in square parcel.

118. Proposed by J. F. TRAVIS, Student in Ohio State University, Columbus, Ohio.

The present worth of a note due January 1, 1896, was \$74,200, when discounted at 4% true discount. The present worth of another note, due July 1, 1896, whose face value was the same as that of the first note, was \$68,900 when discounted at 8% true discount. Find the face of the notes and the date when given, supposing the second note to have been given the same day the first note was. Solve by arithmetic.

I Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $F$=face of each note, $t$=number of years before January 1, 1896, the notes were given.

Then $(1+.04t)(74200)=[(1+.08(t+\frac{1}{4})](68900)$.
$\therefore\ 68900\times .08t=5300(1+.04t)$.
$\therefore\ t=1$ year. $\therefore$ time was January 1, 1895.
$F=(74200)(1.04)=\$77168$.

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Ler $x$=the number of days the first note had to run, and $x+180$ the number the second had to run ; regarding 360 days to the year.

Then we have, since the face of each note was the same,

$$74200\left(1+\frac{4}{100}\frac{x}{360}\right)=68900\left(1+\frac{8}{100}\frac{x+180}{360}\right).$$

Whence, dividing by 100 and transposing,

$$53=\frac{5512x+992160-2968x}{36000}$$

or, $2544x=53(36000)-992160=915840$.

$x=360$ days, or date of each note was January 1st, 1895.

The face value must have been \$74200+the discount, which is 4% of \$74200 or \$2968 ; that is \$77168.

119. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The cost of an article is $\$1\ \frac{9}{.100}$. The selling price is \$1,000. What is the gain per cent.?

Solution by C. C. BEBOUT, Professor of Mathematics, High School, Elgin, Ill., and J. M. HOWIE, Professor of Mathematics, The Nebraska State Normal School, Peru, Neb.

I do not think that the solution of Arithmetic problem 112, published in the August-September MONTHLY is correct. My solution of No. 119 will show wherein I differ. C. C. BEBOUT.

$\$1.\frac{9}{.001}=\$1.+\$\frac{9}{.001}=\$1.+\$9000=\$9001$, the selling price.

If the article costs \$9001., and sells for \$1000, there is no gain, but a loss of \$8001, which is 88.89+% of \$9001.

The question seems to be as to the meaning of the decimal point following the 1 in the expression $1.\frac{9}{.001}$. A common fraction can not have place value, nor can it give place value as a digit in a number. It is simply added to the number to which it is attached and is a fraction of the unit to which it is attached. If it stands alone it is a fraction of the understood (or named) unit. We would not write $3\frac{1}{4}4$ for 354, nor $2\frac{1}{4}$. for 25. Also $4\frac{1}{4}=4+\frac{1}{4}=4.+\frac{1}{4}=4.\frac{1}{4}$. Every in-

teger in our decimal notation is, in theory, followed by a decimal point, and it can make no difference in the meaning of the expression whether the decimal point is expressed or understood. $4.\frac{1}{2}$ must equal 4.5 and not 4.05, which is the equivalent of $4.0\frac{1}{2}$.

So $1.\frac{9}{.001}=1\frac{9}{.001}=1+\frac{9}{.009}$.

Also solved by *COOPER D. SCHMITT, D. A. LEHMAN, ELMER SCHUYLER*, and the *PROPOSER*. These contributors agree that the result is $10\frac{1}{9}\%$. To my mind, the solution and discussion of the problem as published above are correct. ED. F.

## ALGEBRA.

94. Proposed by J. W. YOUNG, Columbus, Ohio.

Solve : $\left[\frac{x^2+14x+1}{p^4+14p^2+1}\right]^3=\frac{x(x-1)^4}{p^2(p^2-1)^4}$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $p^2(p^2-1)^4=A$, $(p^4+14p^2+1)^3=B$.

$$\therefore Ax^6+42Ax^5+591Ax^4+2828Ax^3+591Ax^2+42Ax+A$$
$$=Bx^5-4Bx^4+6Bx^3-4Bx^2+Bx.$$

$$\therefore Ax^6+(42A-B)x^5+(591A+4B)x^4+(2828A-6B)x^3$$
$$+(591A+4B)x^2+(42A-B)x+A=0.$$

$$\therefore A[x^3+(1/x^3)]+(42A-B)[x^2+(1/x^2)]+(591A+4B)[x+(1/x)]$$
$$+(2828A-6B)=0.$$

$$\therefore A[x+(1/x)]^3+(42A-B)[x+(1/x)]^2+(588A+4B)[x+(1/x)]$$
$$+(2744A-4B)=0.$$

Let $[x+(1/x)]=y$.

$$\therefore p^2(p^2-1)^4y^3-(p^{12}+759p^8+2576p^6+759p^4+1)y^2$$
$$+4(p^{12}+189p^{10}+3p^8+3710p^6+3p^4+189p^2+1)y$$
$$-4(p^{12}-644p^{10}+3335p^8-1288p^6+3335p^4-644p^2+1)=0.$$

Let $a=p^2+1/p^2$.

$$\therefore (2-a)^2y^3-(a^3+756a+2576)y^2+4(a^3+189a^2+3332)y$$
$$-4(a^3-644a^2+3332a)=0.$$

$$\therefore (y-a)[(2-a)^2y^2-4(a^2+188a+644)y+4(a^2-644a+3332)]=0.$$

$$\therefore y=a \text{ and } y=\frac{2(a^2+188a+644)\pm 64\sqrt{(a^3+30a^2+252a+392)}}{(2-a)^2}.$$

Let $p+1/p=b$.

$$\therefore y=b^2-2,\ y=\frac{2(b^2+28b+68)}{(b-2)^2},\ y=\frac{2(b^2-28b+68)}{(b+2)^2}.$$

$$\therefore p^2x^2-(p^4+1)x+p^2=0.$$

$$(b-2)^2x^2-2(b^2+28b+68)x+(b-2)^2=0.$$

$$(b+2)^2x^2-2(b^2-28b+68)x+(b+2)^2=0.$$

$$\therefore x=p^2,\ x=1/p^2,\ x=\frac{(p^4+28p^3+70p^2+28p+1)\pm 8(p^3+7p^2+7p+1)\sqrt{p}}{(p-1)^4}.$$

$$x=\frac{(p^4+28p^3+70p^2-28p+1)\pm 8(p^3-7p^2+7p-1)\sqrt{(-p)}}{(p+1)^4}.$$

II. Solution by ELMER SCHUYLER, Reading, Pa.

Let $p^2=t$, $x^2+1=xz$, and $t^2+1=tv$. Then

$$\frac{x^3(z+14)^3}{t^3(v+14)^3}=\frac{x^3(z-2)^3}{t^3(v-2)^3}. \quad \therefore \left(\frac{z+14}{v+14}\right)^3=\left(\frac{z-2}{v-2}\right)^3.$$

By inspection, $z=v$ fulfills conditions.

$$\therefore \frac{x^2+1}{x}=\frac{p^4+1}{p^2}, \text{ which gives } x=p^2 \text{ or } 1/p^2 \text{ (two answers).}$$

$$\text{Since } \frac{(z+14)^3-(v+14)^3}{(v+14)^3}=\frac{(z-2)^3-(v-2)^3}{(v-2)^3},$$

we can eliminate, factor $z-v$, and get a quadratic in $z$, or a quadro-quadratic in $z$, which equations give

$$x=\left[\frac{1+\theta\sqrt{p}}{1-\theta\sqrt{p}}\right]^4,\ \theta^4=1.$$

[Quadro-quadratic meaning quadratic equation whose unknown quantity is a quadratic.]

III. Solution by E. D. ROE, Jr., A. M., Ph. D., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Put $x=z^2$ and divide numerators by $z^6$, and denominators by $p^6$. We get:

$$\left(\frac{z^2+14+(1/z^2)}{p^2+14+(1/p^2)}\right)^3=\frac{[z-(1/z)]^4}{[p-(1/p)]^4}\ldots\ldots(1).$$

We notice that $z=p$, and $1/p$.

Put $z-1/z=y$, $p-1/p=r$, and (1) becomes

$$\left(\frac{y^2+16}{r^2+16}\right)^3=\left(\frac{y}{r}\right)^4 \ldots\ldots(2).$$

Multiplying out, we have,

$$y^6r^4+3.16y^4r^4+3.16^2y^2r^4+16^3r^4-r^6y^4-3.16r^4y^4-3.16^2r^2y^4-16^3y^4=0, \text{ or}$$

$$r^4y^4(y^2-r^2)-3.16^2r^2y^2(y^2-r^2)-16^3(y^2+r^2)(y^2-r^2)=0,$$

$$\text{or } (y^2-r^2)[r^4y^4-16^2(3r^2+16)y^2-16^3r^2]=0 \ldots\ldots(3).$$

From this we have $y^2-r^2=0\ldots\ldots(4)$,

$$\text{or } r^4y^4-16^2(3r^2+16)y^2-16^3r^2=0\ldots\ldots(5).$$

By replacing $y$, and $r$, (4) gives

$$z^2-2+(1/z^2)=x-2+(1/x)=p^2-2+(1/p^2), \text{ or } x+(1/x)=p^2+(1/p^2),$$

whence $x^2-[p^2+(1/p^2)]x+1=0$, and $(x-p^2)[x-(1/p^2)]=0$,

and $x=p^2$, $x=1/p^2$, as before noticed.

Solving (5) for $y^2$, we have

$$y^2=\frac{2.8^2(3r^2+16)\pm 8^2\sqrt{[4(3r^2+16)^2+r^6]}}{r^4}=x-2+1/x, \text{ and } x+1/x=$$

$$=\frac{2(p^2-1)^4+2.8^2[3(p^2-1)^2+16p^2]p^2\pm 8^2p\sqrt{4[3(p^2-1)^2+16p^2]p^2+(p^2-1)^4}}{(p^2-1)^4}$$

The expression under the radical sign is found to be the square of $p^6+15p^4+15p^2+1$. Taking the upper sign and dividing out the factor $(p+1)^4$ from numerator and denominator,

$$x+1/x=\frac{2(p^4+28p^3+70p^2+28p+1)}{(1-p)^4}.$$

Taking the lower sign and similarly dividing out the factor $(p-1)^4$,

$$x+1/x=\frac{2(p^4-28p^3+70p^2-28p+1)}{(1+p)^4}.$$

It is seen that both these values of $x+1/x$ are contained in the formula,

$$\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon^2 p)^4},$$

where $\varepsilon$ is a fourth root of unity, as only even powers of $\varepsilon$ and $\sqrt{p}$ occur, and every even power of $\varepsilon$ is $+1$ or $-1$. For $\varepsilon=+1$, or $-1$, we get the first value, and for $\varepsilon=+i$, or $-i$, we get the second value of $x+1/x$, and these are the only values which the formula admits. Thus

$$x+1/x=\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon^2 p)^4}=\frac{(1+\varepsilon\sqrt{p})^8+(1-\varepsilon\sqrt{p})^8}{(1-\varepsilon\sqrt{p})^4(1+\varepsilon\sqrt{p})^2}$$

$$=\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4+\left(\frac{1+\varepsilon\sqrt{p}}{1+\varepsilon\sqrt{p}}\right)^4=\lambda+1/\lambda$$

if $\lambda$ denote $\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4$. Then as before

$$x^2-[\lambda+(1/\lambda)]x+1=(x-\lambda)[x-(1/\lambda)]=0,$$

and $x=\lambda$, or $x=1/\lambda$, but when $\varepsilon$ takes its four values, the group of values represented by $\lambda$, is the same as the group represented by $1/\lambda$, though not corresponding value by value for the same value of $\varepsilon$. The values of $\lambda$ and $1/\lambda$ are the same for 1, and $-1$, $-1$ and 1, $i$ and $-i$, $-i$ and $i$, respectively.

The six values of $x$ are

$$x=p^2,\ x=1/p^2,\ \text{and either } x=\left(\frac{1+\varepsilon\sqrt{p}}{1-\varepsilon\sqrt{p}}\right)^4,\ \text{or } x=\left(\frac{1-\varepsilon\sqrt{p}}{1+\varepsilon\sqrt{p}}\right)^4$$

for the last four values, or other similar expressions containing $\varepsilon$, which are easily formed.

## GEOMETRY.

**119. Proposed by WILLIAM HOOVER., A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

A sphere touches each of two straight lines which are inclined to each other at a right angle but do not meet; show that the locus of its center is an hyperbolic paraboloid.

**I. Solution by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.**

If the common perpendicular to the two given lines be taken as the $Z$-axis, its length represented by $2c$, its middle point be taken as origin, a plane parallel to the two lines as $xy$-plane, and the projections of the lines on the $xy$-plane as the $X$ and $Y$-axis, respectively, the equations of the lines will be $y=0$, $z=c$, and $x=0$, $z=-c$. The square of the distance of the point $(x', y', z')$, any point in the required locus, from the first line is $y'^2+(c-z')^2$, and from the second

line $x'^2+(c+z')^2$. These distances are equal, and we get $x^2-y^2+4cz=0$ as the required equation. This is the equation of a hyperbolic paraboloid.

A more general problem is to find the locus of points equi-distant from any two non-intersecting straight lines in space.

If the axes are taken as above, except that the bisectors of the angles formed by the projections of the given lines on the $xy$-plane are taken as the $X$ and $Y$-axis, the resulting equation will be

$$\frac{mxy}{1+m^2}+cz=0,$$

which is also the equation of a hyperbolic paraboloid.

II. Solution by the PROPOSER.

With coördinate axes rectangular, the two given lines may be taken as

$$y=mx,\ z=c.....(1),\text{ and }\ y=-mx,\ z=-c.....(2).$$

Let the sphere be $(x-x')^2+(y-y')^2+(z-z')^2=r^2.....(3)$.

(1) intersects (3) where

$$(1+m^2)x^2-2(x'+my')x+(x'^2+y'^2+c^2-2cz'+z'^2-r^2)=0.....(4).$$

(1) will then be tangent to (3) if

$$(x'+my')^2=(1+m^2)(x'^2+y'^2+c^2-2cz'+z'^2-r^2).....(5).$$

Similarly, (2) will be tangent to (3) if

$$(x'-my')^2=(1+m^2)(x'^2+y'^2+c^2+2cz'+z'^2-r^2).....(6).$$

(5)−(6) gives, $mx'y'=-c(1+m^2)z'.....(7)$, the required locus of the center $(x', y', z')$ of (3). But (7) is an hyperbolic paraboloid.

III. Solution by J. W. YOUNG, Graduate Student, Ohio State University, Columbus, Ohio.

Let the two straight lines be

$$\frac{x}{0}=\frac{y}{0}=\frac{z}{1};\quad \frac{x-a}{0}=\frac{y}{1}=\frac{z}{0}.$$

The center of a sphere, which touches these two straight lines will always be equidistant from them. Hence equating distances, putting $(x_1, y_1, z_1)$ for the center, we have

$$x_1^2+y_1^2=(x_1-a)^2+z_1^2.$$

Whence the required locus is easily seen to be

$$y^2-z^2=-2ax+a^2,$$

a hyperbolic paraboloid.

Excellent solutions were received from *G. B. M. ZERR*, and *J. SCHEFFER*.

120. Proposed by P. C. CULLEN, Principal of Public Schools, Indianola, Neb.

Draw a circle tangent to a given circle and tangent to a given chord at a given point.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and HENRY HEATON, A. M., Atlantic, Iowa.

Let $O$ be the center of the given circle, and $P$ the given point in the given chord $AB$.

Through $P$ draw $EF$ perpendicular to chord $AB$. Draw the diameter $COD$ parallel to $EF$.

Through $C$ and $P$ draw line $CH$ terminating at $H$ in the circumference of the given circle.

Draw $OH$ intersecting $EF$ at $M$.

Then will $M$ be the center of a circle tangent to chord $AB$ at $P$, and tangent to the given circle at $H$.

PROOF. The center of a circle tangent to $AB$ at $P$ must lie in the perpendicular $EF$.

The radii $OH$ and $MH$ are drawn to the point of tangency of the two circles. Therefore, the centers $O$ and $M$ and the point of tangency $H$ must lie in the same straight line.

There remains to be proved $MP=MH$.

By construction, $MP$ is parallel to $OC$, and $OC=OH$. Whence $\triangle COH$ and $\triangle PMH$ are similar.

$\therefore$ $MP=MH$, and $M$ is the center of the required circle, $MP$ and $MH$ being radii thereof.

By a similar construction, we find $N$ the center of a tangent circle on the other side of chord $AB$, the point of tangency being $G$.

II. Solution by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.; JOHN J. QUINN, Instructor in Mathematics, Rochester Athenaum and Mechanics Institute, Rochester, N. Y.; GAYLOR CAMERON, Student Heidelberg University, Tiffin, Ohio; and P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.

Suppose the problem solved, and let $O$ be the given circle, and $E$ the center of the required circle, and $C$ the given point. One locus of $E$ is the perpendicular to $AB$ at $C$. From $C$, on this perpendicular, take $CD$=radius of given circle; then $EO=ED$. Hence another locus of $E$ is $HE$, the perpendicular bisector of $OE$. The intersection of $HE$ and $CD$ determines $E$.

III. Solution by the PROPOSER.

Let $AOB$ be the given circle, $AB$ the chord, and $P$ the given point, $C$ the center of given circle.

At $P$ erect perpendicular, and with $CB$ as radius construct circle $MPN$ tangent to $AB$ at $P$. Draw $MN$ intersecting perpendicular at $C$, which is center of the required circle.

Excellent solutions were received from *J. W. YOUNG, J. SCHEFFER, ELMER SCHUYLER, CHAS. C. CROSS, WALTER H. DRANE, ALOIS F. KOVARIK*, and *P. H. PHILBRICK*.

121. Proposed by AUGUSTUS J. REEF, Student in Illinois State Normal University, Carbondale, Ill.

Construct a triangle having given its three medians. [From Wentworth's *Plane and Solid Geometry.*]

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; ALOIS F. KOVARIK, Instructor in Mathematics, Decorah Institute, Decorah, Ia.; CHAS. C. CROSS, Whaleyville, Va.; and the PROPOSER.

*Each median intersects the other medians at a common point two-thirds of the distance from the vertex to the middle of the opposite side.*

Let $AF$, $BD$, and $CE$ be the three medians of a triangle.

*Trisect each of the medians.*

Take any point $O$ as a center, and with a radius equal to two-thirds of $CE$, the greatest median, describe the semi-circumference $HCG$.

Draw the diameter $HOG$.

With a radius equal to two-thirds of $BD$, the next largest median, and $O$ as a center, intersect $HOG$ at $B$.

Bisect, respectively, $HB$ at $N$, $BG$ at $M$, and $MN$ at $P$.

Then, with $NP$, $=PM$, as a radius and $P$ as a center, draw the indefinite arc $NIFM$. *This arc bisects any straight line drawn from point B to outer arc.*

With a radius equal to one-third of median $AF$ and $O$ as a center, intersect arc $NIFM$ at $F$; and with a radius equal to two-thirds of $AF$ and $O$ as a center, describe the indefinite arc $AR$.

Through $F$ and $O$ draw line $FA$ terminating in arc $AR$. Also draw lines $BA$ and $AC$.

Then will $ABC$ be the required triangle.

II. Solution by HENRY HEATON, M. Sc., Atlantic, Ia.; ELMER SCHUYLER, Reading, Pa.; J. D. CRAIG, A. B., New Germantown, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; GAYLOR CAMERON, Tiffin, O.; and W. H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Construct the triangle $DEB$ such that $DE$, $DB$, and $EB$ shall be, respectively, equal to two-thirds the given medians from the angles $A$, $B$, and $C$, of the required triangle.

Draw $EC$ parallel to $BD$, and $DC$ parallel to $BE$, meeting in $C$.

Prolong $ED$ to $A$, making $DA=ED$. Join $AB$ and $AC$.

Then will $ABC$ be the the required triangle.

The demonstration is obvious.

III. Solution by J. W. YOUNG, Columbus, Ohio, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Construct a parallelogram such that $MN$, $MO$, and $MB$ shall be double the given medians.

Draw the other diagonal $NO$. Trisect $NO$ in $A$, $C$. The triangle $ABC$ is the one required, since $PB$ is evidently one of the medians given, and the other medians $QC$ and $AR$ are, respectively, equal to $\frac{1}{3}OB$ and $\frac{1}{3}NB$. This is clear, from the considerations of the similar triangles $AOB$ and $AQC$ ($AQ=\frac{1}{3}AB$, $AC=\frac{1}{3}AO$, $\therefore QC=\frac{1}{3}OB$), and $NCB$ and $ACR$ ($AC=\frac{1}{3}NC$, $RC=\frac{1}{3}BC$, $\therefore AR=\frac{1}{3}NB$).

## CALCULUS.

**90. Proposed by ELMER SCHUYLER, Reading, Pa.**

Prove that the evolute of the logarithmic spiral is an equal logarithmic spiral. [From Byerly's *Integral Calculus.*]

**I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; GEORGE LILLEY, Ph. D., L. L. D., Professor of Mathematics, State University, Eugene, Ore.; WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.; and ELMER SCHUYLER, Reading, Pa.**

The intrinsic equation to the logarithmic spiral $s=k(c^t-1)$.

$ds/dt=kc^t\log c$, for the evolute $\sigma=\pm(ds/dt)]_0^t$.

$\therefore \sigma=kc^t\log c-k\log c=k\log c(c^t-t)$.

$\therefore \sigma=k'(c^t-1)$, an equal spiral.

**II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; and COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.**

Let $P$ be a point of the given curve $r=a^\theta$, $O$ the center of curvature, $PQ$ a tangent at $P$, $PO=\rho$, $SQ=p$=the perpendicular from $S$ upon $PQ$, $SP=r$, $SO=r'$, $SM$ perpendicular to $OP$ and$=p'$.

The pedal equation of the given curve $r=a^\theta$ is $r=p\sqrt{[1+(\log a)^2]}$; we also have $r'^2=\rho^2+r^2-2\rho p$, but $\rho=r\sqrt{[1+(\log a)^2]}$.

$\therefore r'=r\log a$, and since $p'^2=r^2-p^2$, we have

$$p'^2=\frac{r^2(\log a)^2}{1+(\log a)^2}. \quad \therefore p'=\frac{r\log a}{\sqrt{[1+(\log a)^2]}}. \quad \therefore p'=\frac{r'}{\sqrt{[1+(\log a)^2]}},$$

or, $r'=p'\sqrt{[1+(\log a)^2]}$, which is the pedal equation of the evolute and exactly like the pedal equation of the logarithmic spiril.

**III. Solution by CHAS. E. MYERS, Canton, Ohio; and P. H. PHILBRICK, M. S., C. E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles, La.**

Let $r$=the radius vector of the given curve, $p$=the perpendicular on the tangent, $r_1$=the radius vector of the evolute, $p_1$=the perpendicular on its tangent, and $R$=the radius of curvature.

We have for the curve, $r=cp$.....(1).

$\therefore\ dr/dp=c$, and $R=cr$.....(2).

$$r^2=p^2+p_1^2 \quad .....(3). \qquad r_1^2=R^2+r^2-2Rp.....(4).$$

From (1), (2), and (4), we have,

$$r_1^2=c^2p^2(1+c^2)-2c^2p^2=c^2p^2(c^2-1).....(5).$$

From (3), and (1),

$$c^2p^2=p^2+p_1^2, \text{ or, } p^2(c^2-1)=p_1^2.....(6).$$

From (5), and (6),

$$r_1^2=c^2p_1^2, \text{ or, } r_1=cp_1,$$

the equation of a similar and equal logarithmic spiral.

IV. Solution by J. W. YOUNG, Graduate Student, Ohio State University, Columbus, Ohio.

The equation to the logarithmic spiral is

$$r=ae^{m\theta}.$$

The equation to the normal at the point $(r_1, \theta_1)$ on the spiral is

$$r\cos[\theta-(\theta_1+\phi)]=r_1\cos\phi.....(1),$$

where $\phi$ is the angle between $r_1$ and the tangent, and $\phi$ is constant, a known property of the spiral.

Expanding the left-hand member of (1), we have

$$r[\cos\theta(\cos\theta_1\cos\phi-\sin\theta_1\sin\phi)+\sin\theta(\sin\theta_1\cos\phi+\cos\theta_1\sin\phi)=r_1\cos\phi.$$

Dividing through by $\cos\phi$, and remembering that $\tan\phi=$constant, which can easily be shown to be $1/m$, we have

$$r\{\cos\theta[\cos\theta_1-(1/m)\sin\theta_1]+\sin\theta[\sin\theta_1+(1/m)\cos\theta_1]=r_1=ae^{m\theta_1}.$$

The required evolute is the envelope of this line where $\theta_1$ is the parameter. Hence, arranging in terms of functions of $\theta_1$, we have

$$r\{\cos\theta_1[\cos\theta+(1/m)\sin\theta]+\sin\theta_1[\sin\theta-(1/m)\cos\theta]\}=ae^{m\theta}.....(2).$$

Differentiating with respect to $\theta_1$,

$$r\{-\sin\theta_1[\cos\theta+(1/m)\sin\theta]+\cos\theta_1[\sin\theta-(1/m)\cos\theta]\}=ame^{m\theta_1}.....(3).$$

We must eliminate $\theta_1$ between (2) and (3).

Dividing (2) by (3), we have

$$\frac{\cos\theta_1(m\cos\theta+\sin\theta)+\sin\theta_1(m\sin\theta-\cos\theta)}{-\sin\theta_1(m\cos\theta+\sin\theta)+\cos\theta_1(m\sin\theta-\cos\theta)}=\frac{1}{m}.$$

Reducing, $\cos\theta_1.\cos\theta=-\sin\theta_1\sin\theta$, or

$$\frac{\cos\theta_1}{\sin\theta}=-\frac{\sin\theta_1}{\cos\theta}=\frac{\sqrt{(\cos^2\theta_1+\sin^2\theta_1)}}{\sqrt{(\sin^2\theta+\cos^2\theta)}}=1.$$

Hence, $\cos\theta_1=\sin\theta$, $\sin\theta_1=-\cos\theta$.

$\therefore\ \theta_1=\theta-\frac{1}{2}\pi$.

$\therefore$ The required equation is $r=ae^{m(\theta-\frac{1}{2}\pi)}=a'e^{m\theta}$.

V. Solution by P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.

Let $P$ and $Q$ be two points on the curve, $C$ its pole, $PB$, $QB$, the normals at $P$ and $Q$; join $CB$. Then the angles $CPB$ and $CQB$ are equal, and consequently the four points $C$, $P$, $Q$, $B$, lie on a circle. Hence $\angle QCB=\angle QPB$; but in the limit when $P$ and $Q$ are coincident, the angle $QPB$ becomes a right angle and $B$ becomes the center of curvature belonging to point $P$; hence $PCB$ also becomes a right angle, and the point $B$ is determined.

Again, $\angle CBP=\angle CQP$; but, in the limit, the angle $CQP$ is constant; therefore angle $CBP$ is also constant, and since the line $BP$ is a tangent to the evolute at $B$, it follows that the tangent makes a constant angle with the radius vector $CB$. From this property it follows that the evolute in question is another logarithmic spiral.

Again, as the constant angle is the same for the curve and for its evolute, it follows that the latter curve is the same spiral.

Professors M. C. Stevens and Elmer Schuyler refer to the solution in *Williamson's Differential Calculus*, page 300.

## MECHANICS.

### Note on Problem 82.

Professor Zerr, in his solution of this problem, assumed limiting friction at all points. Of course there is limiting friction between box and floor. But limiting friction does not in general exist at both the other two points; for suppose it did, then the sphere in its descent will not revolve about its center.

Hence, taking moments about the center (using his notation) we have

$$\tfrac{1}{3}W'=\frac{2W'}{3\cos\theta+9\sin\theta}.$$

Whence $\theta=60°$, no matter what the relation between $W$ and $W'$. Of

course $W$ and $W'$ might be so related that this would be true, but in general they are not. To see at which point there is limiting friction, we might proceed as follows: Solve on the supposition that limiting friction exists between sphere and box, and not between sphere and wall; then solve with the opposite assumption; one of these results will be found smaller than the other, I think it is the latter; we infer then that limiting friction must exist at wall and not at box, and hence that the latter result is the correct one. W. H. DRANE.

88. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics, Decorah Institute, Decorah, Ia.

Show that the equation to the trajectory is

$$y = x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha},$$

and that $v$ and $a$ can be varied at pleasure, the projectile can in general be made to traverse any two given points in the same vertical plane with the point of projection. [Ex. 83, page 244, Deschanel's *Natural Philosophy*, Part I.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; B. F. SINE, Principal of Normal School, Rock Enon Bridge, W. Va.; ELMER SCHUYLER, Reading, Pa.; and the PROPOSER.

Let $v$=velocity of projection, $\alpha$=angle of elevation, $t$=time, $(x, y)$ the coördinates of the point in its path at the time $t$.

$\therefore$ $x=vt\cos\alpha$=horizontal motion. $y=vt\sin\alpha-\frac{1}{2}gt^2$=vertical motion.

Eliminating $t$, we get at once,

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha}\ldots\ldots(1).$$

(1) is true no matter what be the values of $v$ and $\alpha$.

Let $(m, n)$, $(b, c)$ be the coördinates of two points. Then from (1) we get

$$n=m\tan\alpha-\frac{gm^2}{2v^2\cos^2\alpha}, \qquad c=b\tan\alpha-\frac{gb^2}{2v^2\cos^2\alpha}.$$

$$\therefore\ \alpha=\tan^{-1}\left(\frac{nb^2-m^2c}{mb^2-m^2b}\right), \qquad v^2=\frac{g[(mb^2-m^2b)^2+(nb^2-m^2c)^2]}{2(mb^2-m^2b)(bn-mc)}.$$

These values of $\alpha$ and $v$ will cause the trajectory to pass through the two given points.

89. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Assuming that the Northern Pacific R. R. tracks between Fargo and Bismark (North Dakota) to lie on the 47th parallel of latitude; also that the Limited Express weighs 300 tons, and that a speed of 60 miles per hour is maintained between the two places · find the difference between the vertical pressures on the rails of the Express east and the express west.

I. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Centrifugal force$=mv^2/\rho_1$.

Taking $g=32$ we have $m=18750$ for mass of train.

For 47th parallel of latitude $\rho=14403840$ feet, and $v=1048.89$ feet per second is velocity of a point at that parallel.

The velocity of the train on the surface of the earth is 88 feet per second.

$\therefore$ $1048.89-88=960.89$ feet is velocity in space of train going west.

$1048.89+88=1136.89$ feet is velocity in space of train going east.

For train going west we have for centrifugal force,

$$\frac{mv^2}{\rho}=\frac{18750(960.89)^2}{14403840}=1201.1059 \text{ pounds.}$$

For train going east,

$$\frac{mv^2}{\rho}=\frac{18750(1136.89)^2}{14403840}=1682.1923 \text{ pounds.}$$

$1682.1923-1201.1059=481.0864$ pounds, amount by which the decrease in weight of train going east exceeds the decrease in weight of train going west.

If by vertical pressure is meant that towards center of circle of latitude, the above is the excess in pressure of train going west over that of train going east.

If the pressure is understood to act towards the center of the earth the difference is $(481.0864)\cos 47°=328.6312$ pounds.

If however, the difference in the weight of the two trains is required, we must proceed as follows:

We have assumed the earth to be a perfect sphere. On account of the motion of the earth the apparent line of weight is slightly deflected from the vertical. Let $PR$ be the line of action of apparent weight, $R=mg$ at a point $P$ on the earth's surface. Let $G$ be the value gravity would have if the earth were still, and then the force along $PO$ will be $mG$. Let $\varphi$ be the complement of the latitude of $P$. Resolving along $CP$ and perpendicular to it we get

$$\frac{mv^2}{\rho}=mG\sin\varphi - mg\cos\theta \ldots\ldots (1).$$

$$mg\sin\theta=mG\cos\varphi \ldots\ldots (2).$$

Eliminating $\theta$ from (1) and (2) and solving for $G$ gives

$$G=\frac{v^2\sin\varphi}{\rho}\pm\sqrt{g^2-\frac{v^4}{\rho^2}\cos^2\varphi}.$$

Substituting values for the case in hand we find $G=32.05209$.

From (2),

$$\sin\theta = \frac{G\cos\varphi}{g} = \frac{32.05209 \times .731}{32} = .73219.$$

$\therefore$ $\theta = 47^\circ\ 4'\ 15''$.

$\therefore$ Difference in apparent weight $= 481.0864\cos(47^\circ\ 4'\ 15'') = 327.6583$ lbs.

II Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

This problem depends on the velocities in space of the Express east and the Express west.

In the figure, let $P$ be the point of the train on the 47th parallel, $O$ the center of the earth, $AP$ the normal at $P$, $\varphi$ the angle $PO$ makes with the equatorial diameter $B'x$, $\theta$ the angle the normal makes with the same line, ($\theta$ is the latitude of $P$).

Let $OB=\rho$, $OP=r$, 300 tons $=W$, $f$=centrifugal force in the direction $AP$ (vertical direction), $g$=gravity on 47th parallel, $G$=gravity at equator. $a$=6377377 meters=20923536 feet=equatorial radius, $e$=ellipticity of the earth.

$\rho = r\cos\varphi = a\cos\theta/\sqrt{(1-e^2\sin^2\theta)}$.

Now $\theta = 47^\circ$, $e^2 = .006920928$. $\therefore$ $\rho = 4357445.45$ meters.

One day=86400 seconds.

$\therefore$ $2\pi\rho/86400 = 316.8831$ meters $= 1039.37$ feet per second, the velocity of $P$ due to the earth's rotation.

60 miles an hour=88 feet per second.

1039.37−88=951.37, the train's velocity in space going west.

1039.37+88=1127.37, the train's velocity in space going east.

$f = Wv^2/g\rho = Wv^2/gr\cos\varphi$.

$\therefore$ $F = f\cos\theta = Wv^2\sqrt{(1-e^2\sin^2\theta)}/ag$.

Now $g = G(1+\frac{1}{2}e^2\sin^2\theta)$.* $G = 32.2015235$ feet.

$\therefore$ $g = 32.23130991$ feet per second. $\therefore$ $F = .000000444v^2$ tons.

$\therefore$ $F = .4018399$ tons going west. $F = .5643076$ tons going east.

$\therefore$ Difference$= .1624657$ tons$= 324.9354$ pounds.

*This expression for the true value of gravity in latitude $\theta$ is new to me. If any reader of the MONTHLY can tell me where to find it used previously, and by whom, I would be greatly pleased to know. I believe it to be new and unused before.

## AVERAGE AND PROBABILITY.

76. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

In a given ellipse, the extremities of a focal chord are joined with the center. Find the average area of the angle thus formed.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Sciences, Chester High School, Chester, Pa.; GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; J. SCHEFFER, A. M., Hagerstown, Md.; and L. C. WALKER, Instructor in Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

Let $\theta$=the angle the focal chord makes with the major axis.

The length, $l$, of this focal chord

$$=\frac{a(1-e)^2}{1+e\cos\theta}+\frac{a(1-e^2)}{1+e\cos\theta}=\frac{2a(1-e^2)}{1-e^2\cos^2\theta}.$$

The perpendicular distance from the center to this chord$=ae\sin\theta$.

$\therefore$ Area of triangle$=$

$$A=\frac{a^2e(1-e^2)\sin\theta}{1-e^2\cos^2\theta}.$$

I. When the chords are drawn at equal angular intervals,

$$\Delta=\frac{\int_0^{\frac{1}{2}\pi}Ad\theta}{\int_0^{\frac{1}{2}\pi}d\theta}=\frac{2a^2e(1-e^2)}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\theta d\theta}{1-e^2\sin^2\theta}=\frac{a^2(1-e^2)}{\pi}\log\left(\frac{1+e}{1-e}\right).$$

DEAN, DRANE, SCHEFFER, WALKER.

II. When the abscissas of the extremity of the chord are drawn at equal intervals,

$$\Delta=\frac{\int Adx}{\int dx}=\frac{\int_0^{\frac{1}{2}\pi}\frac{A\sin\theta d\theta}{(1-e\cos\theta)^2}}{\int_0^{\frac{1}{2}\pi}\frac{\sin\theta d\theta}{(1-e\cos\theta)^2}}=a^2e(1-e)^2(1+e)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1-e\cos\theta)^3(1+e\cos\theta)}$$

$$=-\tfrac{1}{2}a^2(1-e)^2(1+e)+\tfrac{1}{2}a^2(1-e)^2(1+e)\int_0^{\frac{1}{2}\pi}\frac{(e+\cos\theta)d\theta}{(1-e^2\cos^2\theta)^2}$$

$$=\frac{a^2(1-e)}{8b}\left(\pi a^3e^3-4b(1-e^2)+20a^2\sqrt{1-e^2}+\frac{2a^2}{e}\sin^{-1}e\right).$$

III. When the chord varies with the arc,

$$\Delta=\frac{\int Ads}{\int ds}.$$

ZERR.

77. Proposed by J. OWEN MAHONEY, B.E., M. Sc., Professor of Mathematics and Science, Cooper Training School, Carthage, Tex.; and ELMER SCHUYLER, Reading, Pa.

A and B are two inaccurate mathematicians whose chance of solving a given question correctly is 1/8 and 1/12 respectively; if they obtain the same result, and if it is 1000 to 1 against their making the same mistake, find the chance that the result is correct. [From *Hall and Knight's Algebra.*]

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; COOPER D. SCHMITT, A. M., Ph. D., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and L. C. WALKER, Instructor in Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

The chance that A and B both get the correct result is $\frac{1}{8}\times\frac{1}{12}=\frac{1}{96}$.

The chance that both get the wrong result is $\frac{7}{8}\times\frac{11}{12}=\frac{77}{96}$.

The chance that they both get the *same* wrong result is $\frac{1}{1000}\times\frac{77}{96}=\frac{1}{13.01}=\frac{1}{1248}$.

$\therefore$ The chance that the result is correct : the chance that the result is not correct :: 13 : 1.

$\therefore$ The required chance is $\frac{13}{14}$.

Also the required chance is $(\frac{1}{96})/(\frac{1}{96}+\frac{1}{1248})=\frac{13}{14}$.

78. Proposed by CHAS. E. MYERS, Canton, O.

Two witnesses, A and B, both make the statement that an event happened in a particular way (two ways being possible). Find the probability of the truth of the statement.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $p$, $q$ be the chances that $A$, $B$ speak the truth, respectively.

Then the chance of the truth of the statement is

$$\frac{pq}{pq+(1-p)(1-q)}=c.$$

Now $p=q=\frac{1}{2}$. $\therefore\ c=\frac{1}{4}/[\frac{1}{4}+(1-\frac{1}{2})(1-\frac{1}{2})]=\frac{1}{4}/\frac{1}{2}=\frac{1}{2}$.

## MISCELLANEOUS.

72. Proposed by E. D. ROE, Jr., A. M., Ph. D., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

If $a$, $b$, and $c$ are integers, and

$$\begin{Bmatrix} b,\ c-b,\ c-1 \\ c-a-1 \\ c-a-1 \end{Bmatrix} >0,$$

prove that the sum of the series,

$$1+\frac{a.b}{1.c}+\frac{a(a+b).b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3c(c+1)(c+2)}+\ldots\ldots$$

is equal to

$$\frac{(c-1)!\,(c-a-b-1)!}{(c-a-1)!\,(c-b-1)!}.$$

I. Solution by the PROPOSER.

Let $fx=\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}du.$

If $0<x<1$, $0\leq u\leq 1$, $(1-xu)^{-a}=1+\frac{a}{1}xu+\frac{a(a+1)}{1.2}x^2u^2$

$$+\frac{a(a+1)(a+2)}{1.2.3}x^3u^3+\ldots\ldots$$

and therefore

$$fx=\int_0^1 u^{b-1}(1-u)^{c-b-1}du+\frac{a}{1}x\int_0^1 u^b(1-u)^{c-b-1}du$$

$$+\frac{a(a+1)}{1.2}x^2\int_0^1 u^{b+1}(1-u)^{c-b-1}du+\ldots\ldots$$

$$=B(b,\ c-b)+\frac{a}{1}xB(b+1,\ c-b)+\frac{a(a+1)}{1.2}x^2B(b+2,\ c-b)+\ldots\ldots(1),$$

where $B(m, n)=\int_0^1 u^{m-1}(1-u)^{n-1}du$, is known as the Beta Function or First Eulerian Integral. (Cf. Byerly's *Integral Calculus*, page 109.)

Now we have l. c. page 110,

$$B(m, n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ \text{with}\ \Gamma(n+1)=n\Gamma(n),\ \text{where}\ \Gamma(n)=\int_0^\infty u^{n-1}e^{-u}du,$$

and is known as the Gamma Function or Second Eulerian Integral.

We have therefore,

$$B(m+1,\ n)=\frac{m\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)}=\frac{m}{m+n}B(m, n).$$

If in this formula $m=b$, $n=c-b$, we get $B(b+1,\ c-b)=(b/c)B(b,\ c-b)$.

Similarly,

$$B(b+2,\ c-b)=\frac{b+1}{c+1}B(b+1,\ c-b)=\frac{b(b+1)}{c(c+1)}B(b,\ c-b).$$

$$B(b+3,\ c-b)=\frac{b+2}{c+2}B(b+2,\ c-b)=\frac{b(b+1)(b+2)}{c(c+1)(c+1)}B(b,\ c-b),$$

and by mathematical induction,

$$B(b+n,\ c-b)=\frac{b(b+1).....(b+n)}{c(c+1).....(c+n)}B(b,\ c-b).$$

By substituting these values in (1), we get

$$\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-xu)^{-a}du=B(b,\ c-b)\Big(1+\frac{a.b}{1.c}x+\frac{a(a+1).b(b+1)}{1.2\quad c(c+1)}x^2$$

$$+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\quad c(c+1)(c+2)}x^3+.....\Big)..... (2).$$

Taking the limit of both members of (2) as $x=1$, we have

$$\Big[\int_0^1 u^{b-1}(1-u)^{c-a-b-1}du=\Big]B(b,\ c-a-b)=B(b,\ c-b)\Big[1+\frac{a.b}{1.c}$$

$$+\frac{a(a+1).b(b+1)}{1.2.c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\quad c(c+1)(c+2)}+\ .....\Big]..... (3),$$

$$\text{or}\ \Big[1+\frac{a.b}{1.c}+\frac{a(a+1).b(b+1)}{1.2\quad c(c+1)}+\frac{a(a+1)(a+2).b(b+1)(b+2)}{1.2.3\quad c(c+1)(c+2)}+.....\Big]$$

$$=\frac{B(b,\ c-a-b)}{B(b,\ c-b)}=\frac{\Gamma(b)\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(b)\Gamma(c-b)}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$=\frac{(c-1)!\ (c-a-b-1)!}{(c-a-1)!\ (c-b-1)!}.....(4),$$

if $a$, $b$, and $c$ are integers, and the inequalities stated in the problem are satisfied.

II Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $y_{cx}=1+\frac{ab}{1.c}x+\frac{a[a+1]b[b+1]}{1.2.c[c+1]}x^2+\frac{a[a+1][a+2]b[b+1][b+2]}{1.2.3.c[c+1][c+1]}x^3+....$

$$\therefore\ \frac{dy_{cx}}{dx}=\frac{ab}{1.c}+\frac{a[a+1]b[b+1]}{1.c[c+1]}x+\frac{a[a+1][a+2]b[b+1][b+2]}{1.2.c[c+1][c+2]}x^2+....$$

$$=\frac{ab}{c}\Big[1+\frac{[a+1][b+]}{1.[c+1]}x+\frac{[a+1][a+2]\,b[b+1][b+2]}{1.2.c[c+1][c+2]}x^2+\ ....\Big]$$

$$\frac{d^2y_{cx}}{dx^2}=\frac{a[a+1]b[b+1]}{c.[c+1]}\Big[1+\frac{[a+2][b+2]}{2.[c+2]}x+\ ....\Big]$$

$$\therefore\ x(1-x)\frac{d^2y_{cx}}{dx^2}+\{c-[a+b+1]x\}\frac{dy_{cx}}{dx}=aby_{cx}.....(1).$$

$$y_{(c-1)x}=1+\frac{ab}{1.[c-1]}x+\frac{a[a+1]b[b+1]}{1.2.[c-1]c}x^2+\frac{a[a+1][a+2]\,b[b+1][b+2]}{1.2.3.[c-1]c[c+1]}x^3+\ldots$$

$$y_{cx}-y_{(c-1)x}=-\frac{abx}{c[c-1]}\left[1+\frac{[a+1][b+1]}{1.[c+1]}x+\frac{[a+1][a+2][b+1][b+2]}{1.2.[c+1][c+2]}x^2+\ldots\right]$$

$$\therefore\ y_{(c-1)x}-y_{cx}=\frac{abx}{c[c-1]}\ldots\frac{c}{ab}\frac{dy_{cx}}{dx}=\frac{x}{c-1}\frac{dy_{cx}}{dx}\ldots\ldots(2).$$

If we make $x$=unity in (1) we get

$$\frac{dy_c}{dx}=\frac{aby_c}{c-a-b-1}.\qquad\therefore\ y_{c-1}-y_c=\frac{aby_c}{[c-1][c-a-b-1]}.$$

$$\therefore\ y_{c-1}=\frac{[c-1][c-a-b-1]+ab}{[c-1][c-a-b-1]}y_c=\frac{[c-a-1][c-b-1]}{[c-1][c-a-b-1)}y_c.$$

By symmetry,

$$y_c=\frac{[c-a][c-b]}{c[c-a-b]}y_{c+1},\quad y_{c+1}=\frac{[c-a+1][c-b+1]}{[c+1][c-a-b+1]}y_{c+2}\ldots\ldots(3,\ 4).$$

(4) in (3) gives

$$y_c=\frac{[c-a][c-a+1][c-b][c-b+1]}{c[c+1][c-a-b][c-a-b+1]}y_{c+2}.$$

$$\therefore\ y_c=\frac{[c-a][c-a+1]\ldots[c-a-1+n][c-b][c-b+1]\ldots[c-b-1+n]}{c[c+1]\ldots[c-1+n][c-a-b][c-a-b+1]\ldots[c-a-b-1+n]}y_{c+n}.$$

Let $[c-a-1]=s$, $[c-b-1]=t$, $[c-1]=u$, $[c-a-b-1]=v$. Then

$$c=\frac{\dfrac{[s+1][s+2]\ldots\ldots[s+n]}{1.2.3\ \ldots\ n}\cdot\dfrac{[t+1][t+2]\ldots\ldots[t+n]}{1.2.3\ldots\ n}}{\dfrac{[u+1][u+2]\ldots\ldots[u+n]}{1.2.3\ldots\ldots n}\cdot\dfrac{[v+1][v+2]\ldots\ldots[v+n]}{1.2.3\ \ldots\ldots n}}y_{c+n}$$

$$=\frac{\dfrac{[1+(1/n)][1+(2/n)]\ldots[1+(s/n)]}{1.2.3\ldots\ldots s}\cdot\dfrac{[1+(1/n)][1+(2/n)]\ldots[1+(t/n)]}{1.2.3\ldots\ldots t}}{\dfrac{[1+(1/n)][1+(2/n)]\ldots[1+(u/n)]}{1.2.3\ldots\ldots u}\cdot\dfrac{[1+(1/n)][1+(2/n)]\ldots[1+(v/n)]}{1.2.3\ldots\ldots v}}y_{c+n}.$$

Let $n=\infty$, then $y_{c+n}=1+\dfrac{ab}{c+\infty}+\dfrac{a[a+1]b[b+1)}{1.2[c+\infty][c+\infty+1]}+\ldots\ldots$

$\therefore y_{c+n}=1.$

$$\therefore y_c=\frac{u!\,v!}{s!\,t!}=\frac{[c-1]!\ [c-a-b-1]!}{[c-a-1]!\ [c-b-1]!}.$$

But $y_c=1+\dfrac{ab}{1.c}+\dfrac{a[a+1]b[b+1]}{1.2.c[c+1]}+\ldots\ldots$

Therefore, etc. [See *Forsyth's Differential Equations*, chapter VI, page 185, for treatment of this series.]

## PROBLEMS FOR SOLUTION.

### ARITHMETIC.

112. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Suppose 10% traction stock is 20% better in the market than 5% mining stock; if my income be $500 from each, how much money have I paid for each, the whole investment bringing 6⅔%?

123. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

If $m=2$ cents be the interest on $M=100$ cents for $p=40$ days, find the yearly rate per cent.

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

### ALGEBRA.

111. Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Solve the equation $x(y+z)=a(x+y+z)$, $y(x+z)=b(x+y+z)$, $z(x+y)=c(x+y+z)$.

112. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

In *Hall and Knight's Higher Algebra*, I find the following:

If $a+b+c=0$, then

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2};\quad \frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2};$$

also if $a+b+c+d=0$, then

$$\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^3+b^3+c^3+d^3}{3}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

QUERY. Is there a general law governing such expressions? Investigate.

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

## GEOMETRY.

131. Proposed by J. W. YOUNG, Graduate Student, Ohio State University, Columbus, Ohio.

Prove that $\lambda+\mu\omega+\nu\omega^2$, where $\lambda$, $\mu$, $\nu$ are integers whose sum is $\pm 1$, represents the points of a quilt formed by regular hexagons. $\omega$=primitive cube root of unity. [From *Harkness and Morley's Introduction to Theory of Functions.*]

132. Proposed by ELMER SCHUYLER, Reading, Pa.

To draw a circle to cut two given circles orthogonally.

133. Proposed by P. C. CULLEN, Principal of Public Schools, Indianola, Neb.

If the two bisectors, trisectors, quadrisectors, etc., of the base angles of a triangle are mutually equal, show that the triangle is isosceles.

134. Proposed by J. C. CREGG, A. M., Superintendent of Schools, Brazil, Ind.

If $ABCD$ is a quadrilateral circumscribing a circle, show that the line joining the middle points of the diagonals $AB$, $CD$ passes through the center of the circle.

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

## CALCULUS.

102. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A right cone has its vertex at the focus of a paraboloid of revolution, the axis of the cone perpendicular to the axis of the paraboloid. Find the volume common to both.

103. Proposed by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A park, in the shape of an ellipse whose diameters are 100 and 50 rods, respectively, is surrounded by a wall: one end of a rope, whose length is the circumference of the ellipse, is fastened (outside of the wall) at one end of the longer diameter and the other end at the otoer end of the same diameter. Over how much surface will a horse graze, which is fastened to a ring moving freely on the rope?

*** Solutions of these problems should be sent to J. M. Colaw not later than January 10.

## AVERAGE AND PROBABILITY.

85. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Two points are taken at random in a circle and a chord drawn through them; a point is then taken at random in each segment. Find the average area of the quadrilateral formed by joining the four points.

86. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

Two points are taken at random in a circular annulus formed by two concentric circles. Find the chance that the straight line joining the points will not cut the inner variable circle.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the mean distance of a random point in a sphere from a point, (1) within, (2) without the sphere.

*** Solutions of these problems should be sent to B. F. Finkel not later than January 10.

# EDITORIALS.

Prof. James Pierpont, of Yale University, was elected a member of the French Mathematical Society.

Dr. Artemas Martin, of the United States Coast and Geodetic Survey, at Washington, D. C., has recently been elected a member of the Deutsche Mathematiker-Vereinigung.

At the recent meeting of the Board of Trustees of the Indiana University, Dr. John B. Faught, Instructor in Mathematics, was given the title of Assistant Professor of Mathematics.

Prof. J. H. Tanner, of Cornell University, is writing an elementary algebra for the "Cornell Series," and Prof. James McMahon is writing an elementary geometry for the same series.

We are pleased to state that Dr. Halsted is receiving many congratulatory letters from readers of the MONTHLY who are reading with interest and profit, his contributions to the MONTHLY in Non-Euclidean Geometry.

Through the kindness of Dr. Alexander Macfarlane, we have received for publication a paper on "The Theory of Mathematical Inference," by Prof. G. J. Stokes, of Queen's College, Cork, Ireland. This paper will appear in the January number, Vol. VII, of the MONTHLY.

A number of our readers have asked us to send them the July issue. In answer we will say, take the cover off the June number and you will then have a June-July number. Through error on the part of the printers the statement of a double number was omitted on the cover.

## *BOOKS AND PERIODICALS.*

*The Use of the Slede Rule.* By F. A. Halsey, Associate Editor "American Machinist," Consulting Engineer Rand Drill Co. 16mo. 84 pages. Price, 50 cents. New York: D. Van Nostrand Co.

In this little book the author has set forth very clearly the use of the Slede Rule and has thus rendered valuable service to the practical computer. B. F. F.

*Grammar School Algebra.* By William J. Milne, Ph. D., LL. D., President of New York State Normal College, Albany, N. Y. 154 pages. Price, 50 cents. 1899. New York, Cincinnati, and Chicago: American Book Company.

This book is characterized by the same methods of presentation exemplified in the other books of Dr. Milne's series. It will meet with hearty approval. J. M. C.

*Standard School Algebra.* By George E. Atwood. 432 pages. Price, $1.20. 1898. New York: The Morse Company.

This book is designed for use in high schools, and academies, and advanced classes in grammar schools. The definitions, demonstration of principles, derivation of rules, model solutions, and illustrations, occupy the last half of the book, and the exercises and problems in the first half. This arrangement may commend itself to many, but we fail to see any real advantage to be gained by reducing the first part of the book to a bare collection of exercises and problems. In other respects the book is highly satisfactory. The second part is marked by a clearness and conciseness in definitions, careful demonstration of principles, and an abundance of illustrations and model solutions. The book satisfactorily meets the requirements of what is best in the science and method of teaching elementary algebra at the present time. J. M. C.

*La Mathématique Philosophie*—Enseignement. Par C. A. Laisant, Répétiteur a L'École Polytechnique Docteur és Sciences. 8vo. Cloth, 292 pages. Price, $1.25. Paris: Georges Carré et C. Naud.

In this work the author has considered the philosophy and teaching of mathematics in a way so as to be of service not only to the student of mathematics, but to teachers as well. In No. 4, Vol. V, of the MONTHLY, Dr. Alexander Macfarlane contributed some remarks *in extenso*, from this book. Professor Laisant, in the remarks referred to, says that during the last twenty-five years, few countries have made greater progress in mathematics than the United States. This remark some of our readers considered a jest on the part of the author. But such is not the case. The book is from first to last a most carefully and sincerely written work, intended to be of the highest service to students and teachers of mathematics. The book is divided into three parts: The first part discusses the philosophy of Pure Mathematics; the second part discusses the philosophy of Applied Mathematics; and the third part treats of the Instruction in Mathematics. The first part contains eight chapters, the first of which has to do with Mathematics and its subdivisions; the second with Arithmetic and the Theory of Numbers, or Arithmologie; the third, Algebra; the fourth, the Infinitesimal Calculus; the fifth, Theory of Functions; the sixth, Geometry; the seventh, Analytical Geometry; the eighth, Rational Mechanics. In the second part, chapter 1 is devoted to general considerations; chapter 2, to Applications of the Calculus; chapter 3, to Applications of Geometry; chapter 4, to Applications of Mechanics. In the third part, chapter 1 is devoted to a General View on the Teaching of Mathematics; chapter 2, Teaching of Arithmetic; chapter 3, Teaching of Algebra and the Advanced Calculus; chapter 4, Teaching of Geometry; chapter 5, teaching of Analytical Geometry; chapter 6, Teaching of Mechanics; chapter 7, The Hierarchy of Teaching.

B. F. F.

*A Short Table of Integrals.* Revised Edition. By Benj. Osgood Peirce, Ph. D., Hollis Professor of Mathematics and Natural Philosophy in Harvard University. 8vo. Cloth, 134 pages. Price, $1.00. Boston: Ginn & Co.

This little book in its revised and enlarged form, contains nearly all the integrals commonly needed by students of the elements of the integral calculus in American colleges. A number of pages of auxiliary formulas, involving trigonometric, hyperbolic, and elliptic functions, and useful in transforming and interpreting integrals, have been added, with a few numerical tables in which are given, though in fine type, the four-place logarithms of natural numbers and of the trigonometric functions, the values of the hyperbolic functions, etc. This book will qe found very serviceable to use with any text-book.

J. M. C.

*The United States Sinking-Fund.* By Theodore L. DeLand, Office of the Secretary of the Treasury, Washington. 1899.

We are indebted to Mr. DeLand for a copy of his solution, equation $u_{x+1}-u_x = r_1(a-u_x)+ru_x$, from advance sheets of Vol. II., No. 12, of the *Mathematical Magazine.*

J. M. C.

*Observational Geometry.* By William T. Campbell, A. M., Instructor in Mathematics in the Boston Latin School. With an Introduction by Andrew W. Phillips, Ph. D., Professor of Mathematics in Yale University. Over 300 Illustrations and Diagrams. 8vo., 240 pages. New York and London: Harper & Brothers. 1899.

The reasoning required in this book depends on direct observation and the measurement of geometric figures constructed by the pupils themselves. Part I. treats of elementary forms, beginning with the cube, and introduces at once the ideas of precision and accuracy. Part II. takes up geometric forms in a more minute manner, developing the ideas of arrangement, order, and symmetry. The matter has been skilfully and clearly presented, the illustations are exceedingly helpful, and the book in all its details seems to have been carefully and honestly written. As an introduction to the study of geometry for pupils of the upper grammar grades, it is perhaps the best book that has yet appeared.

J. M. C.

*The Story of the Philippines* and Our New Possessions, Including the Ladrones, Hawaii, Cuba, and Porto Rico. By Murat Halsted. Sold by the Dominion Co., Chicago, Ill.

This book is beautifully illustrated with half-tone engravings from photographs, etchings from special drawings, etc. The book is nicely bound, and there is a great demand for it, it being written by one of our ablest journalists. B. F. F.

The following periodicals have been received: *The American Journal of Mathematics*, October, 1899; *The Educational Times*, October 1, 1899; *Journal de Mathématiques Élémentaires*, 15 October, 1899; *The Monist*, October, 1899; *Bulletin of the American Mathematical Society*, July, 1899; *L'Intermédiaire des Mathématiciens*, Juillet, 1899; *The Kansas University Quarterly*, April, 1899; *The Mathematical Gazette*, June, 1899; *Mathematisch-naturwissenschaftliche Mitteilungen*, Oktober, *Herausgegeben* von Dr. O. Böklen and Dr. E. Wölffing, Stutgart, Germany.

# THE AMERICAN MATHEMATICAL MONTHLY.

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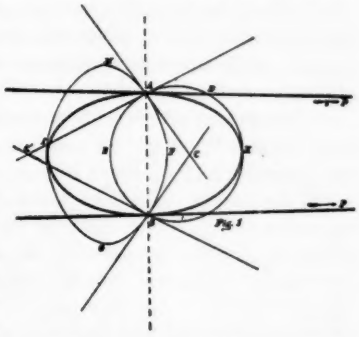
## SOME LOCI AND THEIR PROJECTIONS.


By ARTHUR R. CRATHORNE, B. S., University of Maine.


*If a conic be described through two fixed points, A and B, and touching two given conics which also pass through those points, the locus of the of pole AB with respect to the varying conic is a conic touching the four lines CA, CB, C'A, and C'B, where C and C' are the poles of AB with respect to the two given conics.*

At first reading this proposition or problem seems very complicated and difficult of solution; and, indeed, it would be if the ordinary methods of analytical geometry were used. But treated from the standpoint of projective or the so-called modern geometry, the solution is very easy and leads to some interesting results. This proposition and the one deduced from it later in this article are but examples of a class which lends itself very easily to projective methods.



In figure 1, let $A$ and $B$ be the two given points. Let the two fixed conics, $ADKB$ (called $S$) and $AGIH$ (called $S'$) intersect at the given points. There will then be a doubly infinite number of conics which may pass through the two given points and be tangent to the conics $S$ and $S'$. In figure 1, the heavy lined conic $AKBI$ is one of these conics. (There will be a set which will be tangent at about the points $E$ and $F$). Draw the tangents $AP$

and $BP$. The point of intersection, $P$, will be the pole of the variable conic with respect to the line $AB$. As the conic $AKBI$ takes each of its infinite number of positions, the point $P$ will trace a curve which will touch the tangent lines $CA$, $C'A$, $CB$, and $C'B$.

Let us project $A$ and $B$ into the focoids or circular points at infinity (*Scott's Modern Analytical Geometry*, Art 201). The line $AB$ will be projected into the line at infinity (*Salmon's Conic Sections*, Art. 254). Now since a circle is the only conic which cuts the line at infinity at the focoids, the conics $S$, $S'$ and $AKBI$ must be projected into circles when the points $A$ and $B$ are projected into the focoids (*Scott's Modern Analytical Geometry*, Arts. 117-118). The poles of the line $AB$ will now be at the centers of these circles (*Smith's Conic Sections*, Art. 314). Hence, after projection, the above proposition would read: "The locus of the centers of circles tangent to two given circles is a conic tangent to $CA$, $CB$, $C'A$, and $C'B$."

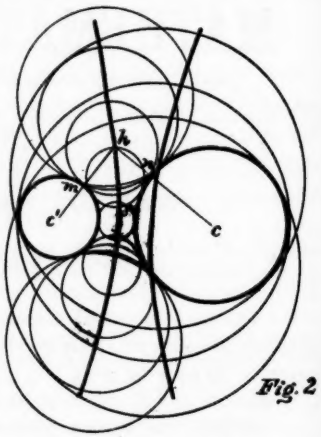


Fig. 2

In figure 2, let the heavy circles be the two given ones (or, the ones into which $S$ and $S'$ are projected). There will be four sets of tangent circles. Figure 2 shows those which lie outside of both the given circles and those which include both circles in their areas. The locus of the centers of these two sets will be an hyperbola, one set making one branch and the second set making the other. The centers of the given circles are the foci of the locus. That this curve is an hyperbola may be easily proved. Let $h$ (Fig. 2) be any point on the curve; let $r$ and $r'$ be the radii of the given circles. Then since $hn=hm$, $hc-hc'=r-r'=$a constant.

From the definition of the curve the locus will be an hyperbola. Moreover this curve will be tangent to the lines connecting $c$ and $c'$ with the focoids or circular points at infinity (*Salmon's Conic Sections*, Art. 258; *Scott's Modern Analytical Geometry*, Art. 129).

Since this property is a descriptive one (*Scott's Modern Analytical Geometry*, Chapter V), it will be true after projection and the only difference between the proposition given at the beginning of this article and the one just proved is in the location of the points $A$ and $B$. In the latter the circular points at infinity are the given points, while in the former any two points may be taken. The former is a general proposition, the latter a special case. In this special case we say "circle" instead of "conic through two fixed points." We say "center" for "pole of the line $AB$ with respect to the conic," and again "hyperbola" for "conic touching the four lines $CA$, $CB$, $C'A$, and $C'B$.

Referring again to figure 1, we see that there may be two cases, one in which the two given conics intersect in real points and the other in which they